

PATTERN EQUIVARIANT COHOMOLOGY AND THEOREMS OF KESTEN AND OREN

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ABSTRACT. In 1966 Harry Kesten settled the Erdős-Szűsz conjecture on the local discrepancy of irrational rotations. His proof made heavy use of continued fractions and Diophantine analysis. In this paper we give a purely topological proof Kesten's theorem (and Oren's generalization of it) using the pattern equivariant cohomology of aperiodic tiling spaces.

1. INTRODUCTION

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and for an irrational number $\xi \in \mathbb{R}$ define a map $T_\xi : \mathbb{T} \rightarrow \mathbb{T}$ by

$$T_\xi(x) = x + \xi \pmod{\mathbb{Z}}$$

It is well known that for any interval $I \subset \mathbb{T}$, the function

$$D(N) = D(N; I) = \#I \cap \{T_\xi^k(x) : 0 \leq k \leq N\} - N \text{Length}(I)$$

is $o(N)$ for each $x \in \mathbb{T}$. That is, $D(N)/N \rightarrow 0$ as $N \rightarrow \infty$ for each x . $D(N)$ is sometimes called the *local discrepancy* or *error function*. As early as the 1920's, it was known (by Hecke and Ostrowski [19, 31]) that if there exists an integer k such that

$$(1.1) \quad \text{Length}(I) \equiv k\xi \pmod{\mathbb{Z}}$$

then $D(N) = O(1)$. That is, $D(N)$ is *bounded*. In the 1960's it was conjectured by Erdős-Szűsz [10] – and proved by Kesten [25] – that the converse holds.

Theorem 1 (Kesten). *Let $I \subset \mathbb{T}$ be an interval, and $\xi \in \mathbb{R}$ be irrational. There exists a constant $C > 0$ such that $|D(N)| < C$ if, and only if, (1.1) holds.*

A generalization of Kesten's theorem for several disjoint intervals is given in the following theorem of Oren [30].

Theorem 2 (Oren). *Let $I_1, \dots, I_L \subset \mathbb{T}$ be L disjoint intervals and $\xi \in \mathbb{R}$ be irrational. There exists a constant $C > 0$ such that $|D(N; I_1) + \dots + D(N; I_L)| < C$ if, and only if, there is a permutation σ such that $b_{\sigma(\ell)} - a_\ell \equiv k_\ell \xi \pmod{\mathbb{Z}}$ for some $k_\ell \in \mathbb{Z}$. Here $I_\ell = [a_\ell, b_\ell]$.*

In this paper we give purely topological proofs of Kesten's and Oren's Theorems using the pattern equivariant cohomology of aperiodic tiling spaces. We reformulate and prove these theorems in the context of “cut-and-project patterns” and the “Bounded Displacement (BD) equivalence relation.” Our proof is based upon a

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recent topological rigidity result for *model sets* [24].

If $S \subset \mathbb{R}^2$ is a strip¹, then we will use the notation

$$S(\mathbb{Z}) = S \cap \mathbb{Z}^2$$

to denote the \mathbb{Z} -points of S . If S has irrational slope, then the projection of $S(\mathbb{Z})$ onto a line parallel to S is called a *cut-and-project set* or a *model set*.² To see the connection between Kesten's result and tilings,³ notice that there is a one-to-one correspondence (given by the projection onto the x -axis) between the \mathbb{Z} -points of the strip⁴

$$S_{\xi, I} = \{(x, y) : \xi x - y - \tilde{x} \in I\}$$

and integers k such that $T_{\xi}^k(\tilde{x}) \in I$. So the discrepancies of the sequence $T_{\xi}(x), T_{\xi}^2(x), \dots$ and the associated cut-and-project set are one and the same.

A current object of interest in tiling theory is the *BD equivalence relation*.⁵ Two subsets $Y_1, Y_2 \subset \mathbb{R}^N$ are said to be BD if there is a bijection $\varphi : Y_1 \rightarrow Y_2$ such that

$$\sup_{\mathbf{y} \in Y_1} \|\mathbf{y} - \varphi(\mathbf{y})\| < \infty.$$

It is not hard to see that a subset of \mathbb{R} is BD to a lattice if and only if its discrepancy (in the sense of §2.2) is bounded. (See [28] for analogous statements in higher dimensions). Kesten's Theorem can then be restated in the language of aperiodic point patterns and the BD equivalence relation as follows:

Theorem 3. *Let Z be a 1 dimensional cut-and-project set obtained from an irrational strip in \mathbb{R}^2 . Z is BD to a lattice if and only if the boundary components of the strip are equivalent mod \mathbb{Z}^2 .*

2. PRELIMINARIES

In this section we will review some of the concepts and results that we will use in the proof of our main results. We will not present this material in generality. The interested reader is encouraged to consult the references for a detailed treatment of the ideas below.

2.1. The topology of cut-and-project patterns. A 2-to-1 cut-and-project pattern Z is a subset of \mathbb{R}^2 obtained from the following construction. Let V and H be transverse lines in \mathbb{R}^2 , and $W \subset H$ (the *window*) be a compact set that is the closure of its interior. Let $\pi_V : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear projection of \mathbb{R}^2 onto V . Then

$$Z = \pi_V((V + W) \cap \mathbb{Z}^2),$$

where $V + W = \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in V, \mathbf{w} \in W\}$. If ∂W has Hausdorff measure zero, then Z is called *regular*. In this paper, W is either an interval or a finite union of

¹For readers who are familiar with cut-and-project tilings, the strip S is simply $S = V + W$ where V is the acting subspace and W is the window.

²Cut-and-project sets can also be obtained from lattices in more than 2 dimensions, but in this paper we only consider those arising from 2-dimensional strips.

³This connection seems to be well known. See for instance [8, 16, 18, 27].

⁴Here \tilde{x} and I are identified with their coset representatives in $[0, 1)$.

⁵See [1, 16, 17, 18, 37, 38] for recent developments and [8, 28, 36] for some earlier developments. BD equivalence is sometimes referred to as *wobbling equivalence* [2, 7].

intervals, so Z is always regular.

Our main tool is the *pattern equivariant* cohomology of Z ([21], or see [34] for a review). We think of Z as the vertices of a tiling of $V \cong \mathbb{R}$ by intervals. We abuse notation by denoting this tiling as Z . A function $f : V \rightarrow \mathbb{R}$ is said to be *strongly* pattern equivariant, or strongly PE, if there exists an $R > 0$ such that for any $\mathbf{v}, \mathbf{v}' \in V$ such that $B_R(\mathbf{v}) \cap Z$ and $B_R(\mathbf{v}') \cap Z$ are translates of each other, we have $f(\mathbf{v}) = f(\mathbf{v}')$. A function is *weakly* PE if it is the uniform limit of strongly PE functions. We can similarly speak of strongly and weakly PE 0-cochains that are evaluated on the vertices of Z and 1-cochains that are evaluated on the edges of Z (and so on for higher-dimensional tilings. In our case the cochain complex ends at dimension 1). The coboundary $d\alpha$ of a (strongly or weakly) PE cochain α is easily seen to be (strongly or weakly) PE.⁶

The cohomologies of the resulting cochain complexes are called the (strong or weak) PE cohomologies of Z , and are denoted $H_s^*(Z)$ and $H_w^*(Z)$. Kellendonk [21] (in a slightly different setting) and Sadun [33] (in this setting) showed that the strong PE cohomology of Z is isomorphic to the Čech cohomology of the associated tiling space, and that the strong PE cohomology with real coefficients is isomorphic to the Čech cohomology with real coefficients. The weak PE cohomology (necessarily with real or complex coefficients) is much more complicated.

By definition, a nontrivial class in $H_s^1(Z)$ can never be represented by the coboundary of a strongly PE 0-cochain. However, it sometimes can be represented by the coboundary of a weakly PE 0-cochain. If so, the class is called *asymptotically negligible* [6, 22], in which case *every* representative of the class is of this form. Let $H_{an}^1(Z) \subset H_s^1(Z)$ denote the asymptotically negligible classes. These classes are described by the following lemma, which is essentially Corollary 4.4 from [23].

Lemma 4. *For a closed strongly PE 1-cochain α , there is a 0-cochain β such that $\alpha = d\beta$. Furthermore, β is weakly pattern equivariant if, and only if, β is bounded.*

We now use a recent result about H_{an}^1 for cut-and-project sets. The following theorem is a special case of a theorem from [24].

Theorem 5. *Let Z be a 2-to-1 dimensional cut-and-project set whose window W is an interval or a finite union of intervals. Then $H_{an}^1(Z)$ is one dimensional and is generated by the differential of the coordinate function on H .*

2.2. Discrepancies and the BD equivalence relation. Given a discrete subset Y of \mathbb{R} , a number $\delta > 0$, and an interval I , we define the *discrepancy* of Y with respect to δ and I to be

$$\text{disc}_Y(I, \delta) = |\#I \cap Y - \delta \text{Length}(I)|$$

If there exists a $\delta > 0$ for which $\text{disc}_Y(I, \delta) = o(\text{Length}(I))$, then one expects $\#I \cap Y \approx \delta \text{Length}(I)$ for large intervals I . If such a number $\delta > 0$ exists, then it is unique and it is called the *density* of Y . Hence with the correct choice of $\delta > 0$ (if it does exist), the discrepancy is a measure of error of the expected number of points of Y in I , versus the true number of points.

The following is a special case of a theorem of Laczkovich [28], and can also be easily proved directly:

⁶We denote the coboundary by d since δ denotes the density of a point pattern.

Theorem 6. *For a discrete subset Y of \mathbb{R} and $\delta > 0$, the following are equivalent:*

- (i) *Y is BD to a lattice of covolume δ^{-1} .*
- (ii) *There exists a constant $c > 0$ such that for every finite interval I*

$$\text{disc}_Y(I, \delta) < c.$$

We can similarly define the discrepancy of any pattern, or of any strongly PE 1-cochain. Every such 1-cochain α can be written as a finite linear combination

$$\alpha = \sum_j c_j \chi(P_j),$$

where the *indicator cochain* $\chi(P_j)$ evaluates to 1 on a particular edge of the pattern P_j and to zero on all other edges. Let

$$\alpha_0 = \sum_j c_j [\chi(P_j) - \delta(P_j)dx],$$

where $\delta(P_j)$ is the density of P_j and dx is the 1-cochain that assigns to each edge its length. (These densities are well-defined thanks to the unique ergodicity of cut-and-project sets.) The discrepancy of α over an interval is α_0 applied to that interval. This is a linear combination of the discrepancies of the patterns P_j .

A cochain α has bounded discrepancy if and only if α_0 has bounded integral, which is if and only if α_0 represents an asymptotically negligible class. Equivalently, α has bounded discrepancy if and only if the cohomology class of α is a linear combination of a class in H_{an}^1 and the class of dx . The following is then an immediate corollary of Theorem 5:

Corollary 7. *Let Z be a 2-to-1 dimensional cut-and-project set whose window W is an interval or a finite union of intervals. Then the set of 1-cohomology classes that are represented by cochains with bounded discrepancy is two dimensional.*

3. PROOF OF THEOREM 3

Let S be an irrational strip, and write $S = V + W$ where V is a one dimensional subspace of \mathbb{R}^2 and $W \subset H$ (a subspace transverse to V) is a closed interval.

Proof. We assume without loss of generality that $\partial S \cap \mathbb{Z}^2$ is empty, since there are at most two points in $\partial S \cap \mathbb{Z}^2$ and these do not affect whether the discrepancy is bounded.

The Čech cohomology of a tiling space \mathcal{T} associated with a non-singular 2-to-1 dimensional cut-and-project set whose window is an interval is well understood [12]. The space is homeomorphic to a “cut torus”, obtained by taking \mathbb{T}^2 , removing a copy of $\pi(\partial S)$, and gluing each point back in twice, once as a limit from one side and once as the limit from the other side. The resulting space has the cohomology of a once- or twice-punctured torus, depending on whether $\pi(\partial S)$ consists of one or two path components. In particular, if the boundaries $\ell_{1,2}$ are related by an element of \mathbb{Z}^2 , then $H_s^1(Z) = \mathbb{R}^2$, while if the boundaries are not related then $H_s^1(Z) = \mathbb{R}^3$, since H^1 of a once- or twice-punctured torus is 2- or 3-dimensional.

Suppose that the two components of ∂S are equivalent (mod \mathbb{Z}^2), and hence that $H_s^1(Z) = \mathbb{R}^2$. By Corollary 7, the cohomology classes of 1-cochains with bounded discrepancy is also 2 dimensional, so *all* classes in H^1 are represented by cochains with bounded discrepancy. Adding the coboundary of a strongly PE 0-cochain to a 1-cochain does not change the boundedness (or unboundedness) of the discrepancy

of that 1-cochain, so in fact all 1-cochains have bounded discrepancy. This shows not only that the cut-and-project set Z has bounded discrepancy (and so is BD to a lattice), but also that *any point pattern Z' locally derived from Z is BD to a lattice*.

If the two components of ∂S are not equivalent, then $H_s^1(Z)$ is strictly larger than the set of 1-cochains with bounded discrepancy, so there exists a strongly PE cochain

$$\alpha = \sum_j c_j \chi(P_j)$$

with unbounded discrepancy. The discrepancy of α is a linear combination of the discrepancies of the indicator cochains $\chi(P_j)$, so at least one of the patterns P_j must have unbounded discrepancy. Let P be such a pattern with unbounded discrepancy.

The indicator cochain χ_P evaluates to 1 on edges whose left endpoints are projections of points in an “acceptance domain” $V + \tilde{W}$, where $\tilde{W} \subset W$. \tilde{W} is obtained by applying the condition that a certain finite set of points must appear in the pattern, and another finite set must not appear. As such, $\tilde{W} + V$ is the intersection of a finite number of translates of S by fixed elements of \mathbb{Z}^2 and a finite number of translates of $\mathbb{R}^2 \setminus S$ by fixed elements of \mathbb{Z}^2 . \tilde{W} can thus be written as a disjoint union of finitely many intervals W_i , each of whose boundary components are related to the boundaries of W by elements of \mathbb{Z}^2 .

Since the multi-strip $\tilde{W} + V$ has unbounded discrepancy, at least one of the strips $\tilde{W}_i + V$ must have unbounded discrepancy. Let $\ell'_{1,2}$ be the boundaries of $\tilde{W}_i + V$. By Theorem 3 with $\pi(\partial S)$ path connected, which we have already proven, ℓ'_1 and ℓ'_2 cannot be equivalent (mod \mathbb{Z}^2). Thus ℓ'_1 must be equivalent to one component ℓ_1 of ∂S and ℓ'_2 must be equivalent to the other component ℓ_2 .

We apply Theorem 3 with $\pi(\partial S)$ path-connected yet again. The strip between ℓ_1 and ℓ'_1 has bounded discrepancy, and the strip between ℓ_2 and ℓ'_2 has bounded discrepancy, and the strip between ℓ'_1 and ℓ'_2 has unbounded discrepancy, so the strip between ℓ_1 and ℓ_2 must have unbounded discrepancy. But that is precisely Z . Since Z has unbounded discrepancy, it cannot be BD to a lattice. \square

4. PROOF OF OREN'S THEOREM

Our proof of Oren's theorem follows the same lines as our proof of Theorem 3. The main idea is to identify generators for the cohomology with unbounded discrepancy (appealing again to Corollary 7 and also to Theorem 3) and observe that the class of the combined intervals yield the trivial class exactly when the hypotheses of Oren's theorem are satisfied.

Proof of Theorem 2. Let S be a disjoint union of strips S_1, \dots, S_L and suppose there are n distinct boundary components of S modulo \mathbb{Z}^2 . In the notation of Theorem 2 the strip S_ℓ is given by

$$S_\ell = \{(t, y) : \xi t - y - x \in I_\ell\}.$$

Let E be the convex hull of S and let \mathcal{T} be the *colored* cut-and-project set obtained in the following way. Color a point $p \in E(\mathbb{Z})$ “ ℓ ” if p belongs to S_ℓ and “ ω ” otherwise. Let \mathcal{T} be the projection of $E(\mathbb{Z})$ onto any line parallel to E .

Keeping in mind that \mathcal{T} is colored, we have $H^1(\mathcal{T}) = \mathbb{R}^{n+1}$ since the associated tiling space of \mathcal{T} is a cut-torus with n cuts. Let H_{ud}^1 be the quotient of $H^1(\mathcal{T})$

by the subspace of classes with bounded discrepancy. By Corollary 7 this quotient space is $(n - 1)$ -dimensional. We will now describe a set of generators for H_{ud}^1 .

Let L_1, \dots, L_n be boundary components that represent each of the \mathbb{Z}^2 classes in ∂S , and let B_j be the convex hull of L_1 and L_j . We claim the classes (in H_{ud}^1) of the indicator cochains i_{B_j} of the B_j 's form a basis for H_{ud}^1 .

To see that these classes span, recall that H^1 is spanned by indicator functions of patterns, and that the acceptance domain of each pattern is a multi-strip whose boundaries are translates (in the vertical direction) by $\mathbb{Z}^2 = \mathbb{Z} + \mathbb{Z}\alpha$ of the various B_j 's. This means that the acceptance domain can be written as an integer linear combination of the i_{B_j} 's, plus (or minus) the indicators of some intervals whose lengths are in $\mathbb{Z} + \mathbb{Z}\alpha$. Since indicator functions of intervals whose lengths are in $\mathbb{Z} + \mathbb{Z}\alpha$ have bounded discrepancy, these do not affect the class in H_{ud}^1 .

Since the $n - 1$ classes of the i_{B_j} 's span H_{ud}^1 , and since H_{ud}^1 is $(n - 1)$ -dimensional, these classes are linearly independent. The only way for a multi-slab to give the zero class is for the boundaries to cancel perfectly mod $\mathbb{Z}^2 = \mathbb{Z} + \alpha\mathbb{Z}$, which is precisely the hypothesis of Oren's theorem. \square

5. CONCLUDING REMARKS

The virtue of pattern equivariant cohomology is that it is not just abstract nonsense—you get to see the cohomology work. In the above proofs, the PE cohomology actually allows you to see what you're counting. This feature (along with some simple observations about the topology of the punctured torus, and some basic linear algebra) yields Kesten's and Oren's theorems without any Diophantine analysis, thereby demonstrating both the power and the intuitive appeal of PE cohomology.

There is a large literature consisting of generalizations and reproofs of Kesten's theorem (see [3, 11, 14, 26, 32, 35] for a small sample), including cohomology-type proofs [20] and [15] using dynamical cocycles on \mathbb{T} . As far as we know, this is the first proof of Kesten's theorem that deals directly with the associated tilings.

We remark on one generalization of Kesten's theorem, the notion of a *bounded remainder set* (BRS). This concept has been studied by a number of authors, such as [11, 14, 18, 29, 32]. Windows that are BRS's yield examples of cut-and-project sets that are BD to lattices, as has been recently reported in [18]. With projections to spaces of dimension higher than one, however, the notion of a bounded remainder set is too strong—one can have windows that are not BRS's but still generate cut-and-project sets that are BD to lattices.

Consider the following reformulation of Kesten's theorem:⁷

For an irrational strip S , $S(\mathbb{Z})$ is BD to a lattice if, and only if, S is the closure of a fundamental domain of a cyclic subgroup of \mathbb{Z}^2 .

For higher dimensional spaces V , we have the following:

Let S be an irrational slab⁸ in \mathbb{R}^N . If S is the closure of a fundamental domain for a cyclic subgroup of \mathbb{Z}^N , then the set $S(\mathbb{Z})$ is BD to a lattice.

⁷This formulation, related ideas, and similar results—especially in identifying the role of fundamental domains—were reported in [8].

⁸We will say that $S \subset \mathbb{R}^N$ is a slab if it is the closed convex hull of two distinct parallel codimension one hyperplanes. It is irrational if its boundary descends to a dense subset of \mathbb{T}^N .

The proof of the above statement follows without much difficulty from the following observation: if L is a cyclic subgroup of \mathbb{Z}^N , then $S(\mathbb{Z})$ can be (modulo some points on the boundary) identified with \mathbb{Z}^N/L . But \mathbb{Z}^N/L is a lattice in the quotient group \mathbb{R}^N/L ! To show that $S(\mathbb{Z})$ is BD to a lattice in \mathbb{R}^N (not just in the quotient) we appeal to simple variant of Proposition 2.1 from [17] (where the group \mathbb{R}^N is replaced with \mathbb{R}^N/L). This argument can also be made for subgroups of \mathbb{Z}^N of higher rank, yielding a non-trivial family of cut-and-project sets that are BD to lattices. However, unlike in the 2-dimensional situation, the converse to the above result is false in general by Theorem 1.2 of [17].

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